Indeterminant forms of limits

$\frac{0}{0}$, $\frac{\infty}{\infty}$, 0^0, 1^∞, $\infty - \infty$,

How do you compute them?

Aside: Tangent line approximation to a function.

$(a, f(a))$ point

$f'(a)$ slope

$$y - f(a) = f'(a)(x-a)$$

$$\boxed{y = f(a) + f'(a)(x-a)}$$

equation of tangent line.

If f is differentiable at a,

then $f(x) \approx f(a) + f'(a)(x-a)$

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l'Hopital's Rule: If a limit

$\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ has the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$,

ie $\left(\lim\limits_{x\to a} f(x) = 0 = \lim\limits_{x\to a} g(x)\right)$ or $\left(\lim\limits_{x\to a} f(x) = \infty = \lim\limits_{x\to a} g(x)\right)$

$\frac{0}{0}$ $\frac{\infty}{\infty}$

then $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ sometimes simpler.

[we assume f & g are differentiable at $x=a$].

Proof: $\left(\frac{0}{0}\text{ case}\right)$ $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(a)+f'(a)(x-a)}{g(a)+g'(a)(x-a)}$

because, as $x\to a$,
$f(x)\longrightarrow f(a)+f'(a)(x-a)$ tangent line approx.

Since the limit has $\frac{0}{0}$ form, $f(a)=0=g(a)$

$\displaystyle\Rightarrow \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(a)\cancel{(x-a)}}{g'(a)\cancel{(x-a)}}$

$\displaystyle= \lim_{x\to a}\frac{f'(x)}{g'(x)}.$ $\blacksquare$ Q.E.D. "Dude, we're done, let's party!"

Example: $\displaystyle\lim_{x\to\infty}\frac{\overset{\to\infty}{x}+xe^{-x}\overset{\to ?}{}}{\underbrace{\sin(x)}_{\text{\textasciitilde}}+\underbrace{\sqrt{x^2+x}}_{\to\infty}}\to\infty$.

$\displaystyle\overset{l'H}{=}\lim_{x\to\infty}\frac{1+e^{-x}-xe^{-x}}{\cos(x)+\frac{1}{2}(x^2+x)^{-1/2}\cdot(2x+1)}$

$$= \lim_{x\to\infty} \frac{1 + e^{-x} - xe^{-x}}{\cos(x) + \frac{1}{2}\frac{(2x+1)}{\sqrt{x^2+x}}}$$

$e^{-x} \to 0$, $xe^{-x} \to ?$, $\frac{\infty}{\infty} \to ?$

Let's look at ? limits:

$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} \quad \frac{\infty}{\infty}$$

$$\overset{l'H}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

$$\lim_{x\to\infty} \frac{2x+1}{\sqrt{x^2+x}} = \lim_{x\to\infty} \frac{\cancel{x}(2+\frac{1}{x})}{\cancel{x}\sqrt{1+\frac{1}{x}}} = 2$$

$$\sqrt{x^2(1+\tfrac{1}{x})} = \sqrt{x^2}\sqrt{1+\tfrac{1}{x}} = x\sqrt{1+\tfrac{1}{x}}$$

Back to limit

$$\lim_{x\to\infty} \frac{x + xe^{-x}}{\sin(x) + \sqrt{x^2+x}} = \lim_{x\to\infty} \frac{\cancel{x}(1+e^{-x})}{\cancel{x}\left(\frac{\sin(x)}{x} + \sqrt{1+\frac{1}{x}}\right)}$$

$x \to \infty$, $xe^{-x} \to 0$, $\to \infty$, $e^{-x} \to 0$

$\sqrt{x^2+x} = x\sqrt{1+\frac{1}{x}}$

$\frac{\sin(x)}{x} \to 0$, $\sqrt{1+\frac{1}{x}} \to 1$

$-1 \le \sin(x) \le 1$

$-\frac{1}{x} \le \frac{\sin(x)}{x} \le \frac{1}{x}$

$$= \lim_{x \to \infty} \frac{1}{1} = \boxed{1}.$$

Example $\displaystyle \lim_{x \to 0} \frac{x \sin(x)}{\cos(x) - 1}$ $\to 0 \cdot 0 = 0$; $\cos(x) \to 1$, $\cos(x) - 1 \to 0$

Method 1 (l'H):

$\frac{0}{0}$ form

$$\lim_{x \to 0} \frac{x \sin(x)}{\cos(x) - 1} \overset{l'H}{=} \lim_{x \to 0} \frac{1 \cdot \sin(x) + x\cos(x)}{-\sin(x)}$$

$$= \lim_{x \to 0} \frac{\sin(x) + x\cos(x)}{-\sin(x)} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$\overset{l'H}{=} \lim_{x \to 0} \frac{\underbrace{\cos(x)}_{\to 1} + 1 \cdot \underbrace{\cos(x)}_{\to 1} + x\underbrace{(-\sin(x))}_{\to 0}}{-\underbrace{\cos(x)}_{\to 1}} = \frac{2}{-1}$$

$$= \boxed{-2}.$$

Check:

Type some Sage code below and press E

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f(x)=x*sin(x)/(cos(x)-1)
show(f(x))
L=limit(f(x),x=0)
show(L)
```

Method 2 (no L'H).

$$\lim_{x\to 0}\frac{x\sin(x)}{\cos(x)-1} = \lim_{x\to 0}\frac{x\sin(x)\left(\cos(x)+1\right)}{\left(\cos(x)-1\right)\left(\cos(x)+1\right)}$$

$\cos^2(x)-1 = -\sin^2(x)$

$$= \lim_{x\to 0}\frac{x\cancel{\sin(x)}\left(\cos(x)+1\right)}{-\sin^2(x)} = \lim_{x\to 0}(-1)\underbrace{\left(\frac{x}{\sin(x)}\right)}_{\downarrow\ 1}\underbrace{\left(\cos(x)+1\right)}_{\downarrow\ 2}$$

$-1\cdot\sin(x)$

Special Limit: $\lim_{x\to 0}\frac{\sin(x)}{x}=1$

$\frac{x}{\sin(x)} \to 1$

$$= \boxed{-2}.$$

Example Find $\lim_{x\to 0}\frac{xe^x - x}{\sin^2(3x)}$

$\frac{0}{0}$ form

$$\overset{l'H}{=} \lim_{x\to 0} \frac{1\cdot e^x + xe^x - 1}{2\sin(3x)\cdot\cos(3x)\cdot 3}$$

$$= \lim_{x\to 0} \frac{e^x + xe^x - 1}{6\sin(3x)\cos(3x)} \qquad \frac{1+0-1}{6\cdot 0\cdot 1} \quad \frac{0}{0} \text{ form}$$

$$\overset{l'H}{=} \lim_{x\to 0} \frac{\underbrace{e^x}_{\to 1} + \underbrace{1\cdot e^x}_{\to 1} + \underbrace{x\cdot e^x}_{\to 0}}{6\underbrace{\cos(3x)}_{1}\cdot 3\cdot\underbrace{\cos(3x)}_{1} + 6\underbrace{\sin(3x)}_{0}\left(-\underbrace{\sin(3x)}_{0}\cdot 3\right)}$$

$$= \frac{2}{18} = \boxed{\frac{1}{9}}$$

$\boxed{\text{Ex}}$ Find $\lim_{t\to\infty} t^2 e^{-3t}$ $\qquad e^{-3t} = \frac{1}{e^{3t}}$

($\infty \cdot 0$ form

$$= \lim_{t\to\infty} \frac{t^2}{e^{3t}} \qquad \frac{\infty}{\infty}$$

$$\overset{l'H}{=} \lim_{t\to\infty} \frac{2t}{3e^{3t}} \qquad \frac{\infty}{\infty}$$

$$\overset{l'H}{=} \lim_{t\to\infty} \frac{2}{\underbrace{9e^{3t}}_{\to\infty}} = \boxed{0}$$

$\boxed{\text{Ex}}$ $\lim_{x\to\infty}\left(\sqrt{x^2+x+1} - x\right)$

$\infty - \infty$

$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+x+1} - x\right)\left(\sqrt{x^2+x+1} + x\right)}{\sqrt{x^2+x+1} + x}$

$= \lim_{x\to\infty} \frac{x^2+x+1 - x^2}{\sqrt{x^2+x+1} + x}$

$= \lim_{x\to\infty} \frac{x+1}{\sqrt{x^2+x+1} + x}$

Could we l'H

$\frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{\cancel{x}\left(1+\frac{1}{x}\right)}{\cancel{x}\left(\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1\right)} = \boxed{\frac{1}{2}}$

1

2

$\sqrt{x^2+x+1} = \sqrt{x^2\left(1+\frac{1}{x}+\frac{1}{x^2}\right)} = \sqrt{x^2}\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}$

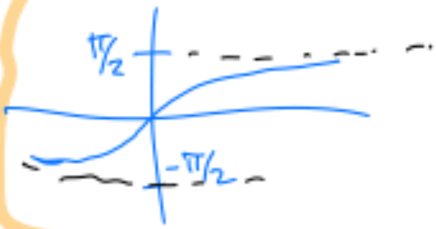
$= x\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}$

Find ① $\lim\limits_{x\to\infty} \left(\frac{x^2+1}{x^2-1}\right)^{3x+x^2}$

$3x+x^2 \to \infty$

$\frac{x^2+1}{x^2-1} \to 1$ $\qquad 1^{\infty}$ form.

② $\lim\limits_{x\to\infty} \arctan(x) = \frac{\pi}{2}$



$\lim\limits_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}$

$\tan\theta \to \infty$ as $\theta \to \frac{\pi}{2}^-$

$\tan\theta \to -\infty$ as $\theta \to -\frac{\pi}{2}^+$

Find ① $\lim\limits_{x\to\infty} \left(\frac{x^2+1}{x^2-1}\right)^{3x+x^2}$

$3x+x^2 \to \infty$

$\frac{x^2+1}{x^2-1} \to 1$ $\qquad 1^{\infty}$ form.

Let

$$L = \lim_{x\to\infty} \left(\frac{x^2+1}{x^2-1}\right)^{3x+x^2}$$

$$\Rightarrow \ln L = \ln\left(\lim_{x\to\infty}\left(\frac{x^2+1}{x^2-1}\right)^{3x+x^2}\right)$$

since $\ln(x)$ is continuous

$$= \lim_{x\to\infty} \ln\left(\frac{x^2+1}{x^2-1}\right)^{3x+x^2}$$

$$\Rightarrow \ln L = \lim_{x\to\infty} \underbrace{(3x+x^2)}_{\infty}\,\underbrace{\ln\underbrace{\left(\frac{x^2+1}{x^2-1}\right)}_{1}}_{0}$$

$\boxed{\ln(A^B) = B\cdot\ln(A)}$

$$\Rightarrow \ln L = \lim_{x\to\infty} \frac{\ln\left(\frac{x^2+1}{x^2-1}\right)}{\frac{1}{3x+x^2}}$$

$$= \lim_{x\to\infty} \frac{\ln(x^2+1)-\ln(x^2-1)}{(3x+x^2)^{-1}} \qquad \frac{0}{0} \text{ form}$$

$\boxed{\ln\left(\frac{A}{B}\right) = \ln(A)-\ln(B)}$

$$\overset{l'H}{=} \lim_{x\to\infty} \frac{\frac{2x}{x^2+1} - \frac{2x}{x^2-1}}{(-1)(3x+x^2)^{-2}\cdot(3+2x)}$$

$$= \lim_{x\to\infty} \frac{\frac{\overset{2x^3-2x}{2x(x^2-1)} - \overset{-2x^3-2x}{2x(x^2+1)}}{(x^2+1)(x^2-1)}}{-\frac{(3+2x)}{(3x+x^2)^2}}$$

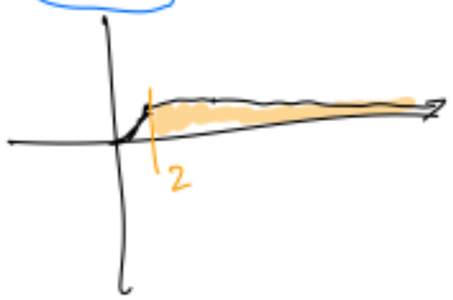
$$= \lim_{x\to\infty} \frac{\frac{-4x}{(x^2+1)(x^2-1)}}{\frac{-(3+2x)}{(3x+x^2)^2}}$$

$$\ln L = \lim_{x\to\infty} \frac{\frac{\cancel{x}(-4)}{\cancel{x^4}(1+\frac{1}{x^2})(1-\frac{1}{x^2})}}{\frac{-\cancel{x}(\frac{3}{x}+2)}{\cancel{x^4}(\frac{3}{x}+1)^2}}$$

$$\Rightarrow \ln L = \frac{\frac{-4}{1}}{\frac{-2}{1}} = 2$$

$$\Rightarrow \boxed{L = e^2}$$

Ex Find $\int_2^\infty x^2 e^{-2x}\,dx$



$$= \lim_{a\to\infty} \int_2^a \underbrace{x^2 e^{-2x}\,dx}$$

parts

$u = x^2 \quad du = 2x\,dx$

$dv = e^{-2x}\,dx \quad v = -\frac{1}{2}e^{-2x}$

$$= \lim_{a\to\infty}\left[-\frac{1}{2}x^2e^{-2x} - \underbrace{\int -\frac{1}{2}e^{-2x}\cdot 2x\,dx}\right]\Bigg|_2^a$$

$$+\underbrace{\int x e^{-2x}\,dx}$$

$u = x \quad du = dx$

$dv = e^{-2x}dx \quad v = -\frac{1}{2}e^{-2x}$

$$= \lim_{a\to\infty}\left[-\frac{1}{2}x^2e^{-2x} + \left[-\frac{x}{2}e^{-2x} - \underbrace{\int -\frac{1}{2}e^{-2x}dx}_{\underbrace{+\frac{1}{2}\int e^{-2x}dx}_{-\frac{1}{4}e^{-2x}}}\right]\right]_2^a$$

$$= \lim_{a\to\infty}\left[-\frac{1}{2}x^2e^{-2x} - \frac{xe^{-2x}}{2} - \frac{1}{4}e^{-2x}\Big|_2^a\right]$$

$$= \lim_{a\to\infty}\left[-\frac{1}{2}a^2e^{-2a} - \frac{a}{2}e^{-2a} - \frac{1}{4}e^{-2a} - \left(-2e^{-4} - e^{-4} - \frac{1}{4}e^{-4}\right)\right]$$

$$= \lim_{a\to\infty}\left[\underbrace{-\frac{1}{2}a^2e^{-2a} - \frac{a}{2}e^{-2a}}_{\text{check} \star} - \frac{1}{4}\underset{\downarrow\, 0}{e^{-2a}} + \frac{13}{4}e^{-4}\right]$$

$$\star \quad \lim_{a\to\infty}\underset{-\infty\,\cdot\,0}{\left(-\frac{1}{2}a^2 - \frac{a}{2}\right)e^{-2a}} = \lim_{a\to\infty}\frac{-1}{2}\underset{\frac{\infty}{\infty}}{\left(\frac{a^2+a}{e^{2a}}\right)}$$

$$\overset{l'H}{=} \lim_{a\to\infty} -\frac{1}{2}\left(\frac{2a+1}{2e^{2a}}\right)$$

$$\overset{l'H}{=} \lim_{a\to\infty} -\frac{1}{2}\left(\frac{2}{4e^{2a}}\right) = \boxed{0}.$$

$$\int_2^\infty x^2 e^{-2x}\,dx = \boxed{\frac{13}{4}e^{-4}} = \boxed{\frac{13}{4e^4}}$$

Sometimes we can't compute an improper integral, but we need to know whether it converges (ie limit is a #) or diverges (limit DNE or $= \pm\infty$).

We will look at ways to tell if an integral converges or diverges. Idea: comparison

$\boxed{\text{Ex}}$ Does $\int_1^\infty \frac{\sin(x)+2}{x^4}\,dx$ converge or diverge?

Observe: $-1 \le \sin(x) \le +1$

$-1+2 \le \sin(x)+2 \le 1+2$

$1 \le \sin(x)+2 \le 3$

$$0 \le \underbrace{\frac{1}{x^4}}_{\text{simple}} \le \underbrace{\frac{\sin(x)+2}{x^4}}_{\text{complicated.}} \le \underbrace{\frac{3}{x^4}}_{\text{simple}}.$$

Thus $0 \le \int_1^\infty \frac{1}{x^4}\,dx \le \int_1^\infty \frac{\sin(x)+2}{x^4}\,dx$

$\le \int_1^\infty \frac{3}{x^4}\,dx$

Note: $\int_1^\infty \frac{1}{x^4}\,dx = \lim_{b\to\infty} \int_1^b x^{-4}\,dx = \lim_{b\to\infty} -\frac{1}{3}x^{-3}\Big|_1^b$

$= \lim_{b\to\infty} \Big(\underbrace{-\frac{1}{3}b^{-3}}_{\to 0} - \left(-\frac{1}{3}\right)(1) \Big] = \frac{1}{3}.$

$$\Rightarrow 0 \le \frac{f}{3} \le \int_1^\infty \frac{\sin(x)+2}{x^4}\,dx$$

$$\le 3\cdot\frac{1}{3} = 1$$

$\therefore$ The integral converges (to a # between $\frac{1}{3}$ & 1.)